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Finite Waiting Space Bulk Service System

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SUMMARY

This paper discusses the ergodic queue length distribution of a bulk service system with finite waiting space by the method of the imbedded Markov chain. The system under consideration is a queuing system with Poisson arrivals, general service times, single server and where service is performed on batches of random size.

1. Introduction

In this paper we analyze a queuing system with Poisson arrivals, general service times, single server with variable batch capacity and finite waiting space. Such a system is denoted by $M/G^{(Y)}/1/(N+1)$ and is usually called a bulk service system. The stationary queue size distribution for this bulk service system is obtained by the method of the imbedded Markov chain. The method is based upon the fact that if the input is of the Poisson type, the length of the queue at epochs just when a batch departs constitutes a Markov chain which is "imbedded" in the continuous time parameter process. For an epoch at which there are no customers remaining to be served, the service is discontinued and the next observation on the process occurs at the end of the service period for the first subsequent arrival. The "imbedded" Markov chain is completely characterized once the transition probability matrix has been determined.

Special cases of the bulk service system considered in this paper have been studied by several authors such as Finch [3], Jain [4], Bhat [1,2], Takamatsu [9], Jaiswal [5], Rao [7], Keilson [6], Singh [8].

2. $M/G^{(Y)}/1/(N+1)$ Queuing System:

This bulk service system can be described as follows:

(i) Customers arrive one at a time in a Poisson process with parameter λ. The probability that j customers arrive in a time interval $(0,t)$ is therefore

$$k_j(t) = e^{-\lambda t}\frac{(\lambda t)^j}{j!}$$

(ii) The customers are served in batches of variable capacity. We assume the service capacity to be s, i.e., not more than s customers can be served at any time. Let $t_1, t_2, \ldots$ be the instants of departure of the successive batches, and denote by v_n the service time of the batch departing at t_n. We assume that $\{v_n\}$ is a sequence of identically distributed independent random variables with a common distribution function $G(t), (0 \leqq t < \infty)$. We further assume that v_n are independent of the arrival process.

(iii) Let $s - Y_n$ be the capacity for service ending at t_{n+1} $(n=0, 1, 2, \ldots)$. We assume that the random variables Y_n are identically distributed and mutually independent, and also independent of the arrival process. Let

$$\begin{aligned} \Pr\{Y_n = r\} &= b_r\,, & 0 \leqq r \leqq s\,, \\ &= 0 & r > s \end{aligned}$$

This may also happen if Y_n customers of the service ending at t_n need to go for another service, the total capacity of service of any batch being s. Then for the service starting after t_n, the server takes min $(s - Y_n$, whole queue length). Let

$$B_j = \sum_{r=0}^{j} b_r = \Pr\{\text{customers already present with the server at a service epoch } \leq j\}$$

and

$$B_s(x) = \sum_{r=0}^{s} b_r x^r .$$

Note that

$$B_s(1) = B_s = \sum_{r=0}^{s} b_r = 1 \text{ and } B_0 = b_0 .$$

(iv) The waiting room has a fixed capacity of $(N+1)$ customers (including those in service). An arrival finding the waiting room full balks and an arrival after joining the system does not renege. The server is never idle in the presence of customers.

3. Analysis of the $M/G^{(Y)}/1/(N+1)$ System:

We say that the system is in state E_j when there are j customers in the system (including the batch just moving to be served). We define the transition probabilities:

$$\gamma_{ij} = \Pr\{\text{next state is } E_j \mid \text{previous state was } E_i\}$$

and the equilibrium probabilities

$$P_j = \Pr\{\text{the system is in state } E_j\}, \qquad j = 0, 1, 2, \ldots, N.$$

We next introduce the generating function

$$P(x) = \sum_{j=0}^{N-1} p_j x^j \tag{1}$$

Now let x_n be the number of customers arrived during a service period ending at t_n; the distribution of x_n is given by

$$\Pr\{X_n = j\} = k_j = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} dG(t) \tag{2}$$

We next define

$$K(X) = \sum_{j=0}^{N-1} k_j x^j \tag{3}$$

The transition probability matrix for this system is given in Table 1.

From the transition probability matrix, we see that the equations determining the equilibrium probabilities are:

$$\begin{aligned} p_j = {} & k_j p_0 + (k_j B_{s-1} + k_{j-1} b_s) p_1 + (k_j B_{s-2} + k_{j-1} b_{s-1} + k_{j-2} b_s) p_2 + \ldots \\ & + (k_j B_1 + k_{j-1} b_2 + \ldots + k_{j-s+1} b_s) p_{s-1} \\ & + (k_j b_0 + k_{j-1} b_1 + \ldots + k_{j-s} b_s) p_s + \ldots \\ & + (k_{j-N+s-1} b_0 + k_{j-N+s} b_1 + \ldots + k_{j-N+1} b_s) p_{N-1} \\ & + (k_{j-N+s} b_0 + k_{j-N+s-1} b_1 + \ldots + k_{j-N} b_s) p_N \\ & \text{for } j = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{4}$$

and

$$\begin{aligned} p_N = {} & l_N p_0 + (l_N B_{s-1} + l_{N-1} b_s) p_1 + \ldots + (l_{s+1} b_0 + l_s b_1 + \ldots l_1 b_s) p_{N-1} \\ & + (l_s b_0 + \ldots + l_0 b_s) p_N , \end{aligned}$$

where

TABLE 1

Transition probability matrix for $M/G^{(Y)}/1/(N+1)$ system

$i \backslash j$	0	1	2		j		$N-1$	N
0	k_0	k_1	k_2		k_j		k_{N-1}	l_N
1	k_0B_{s-1}	$k_1B_{s-1}+k_0b_s$	$k_2B_{s-1}+k_1b_s$	...	$k_jB_{s-1}+k_{j-1}b_s$	...	$k_{N-1}B_{s-1}+k_{N-2}b_s$	$l_NB_{s-1}+l_{N-1}b_s$
2	k_0B_{s-2}	$k_1B_{s-2}+k_0b_{s-1}$	$k_2B_{s-2}+k_1b_{s-1}+k_0b_s$	...	$k_jB_{s-2}+k_{j-1}b_{s-1}+k_{j-2}b_s$	...	$k_{N-1}B_{s-2}+k_{N-2}b_{s-1}+k_{N-3}b_s$	$l_NB_{s-2}+l_{N-1}b_{s-1}$
⋮	⋮	⋮	⋮	...			⋮	⋮
$s-1$	k_0B_1	⋮	⋮	...	$k_jB_1+k_{j-1}b_2+\ldots+k_{j-s+1}b_s$		⋮	⋮
s	k_0b_0	$k_1b_0+k_0b_1$	$k_2b_0+k_1b_1+k_0b_2$	...	$k_jB_0+k_{j-1}b_1+\ldots+k_{j-s}b_s$	...	$k_{n-1}b_0+\ldots+k_{N-s-1}b_s$	$l_Nb_0+l_{N-1}b_1+\ldots+l_{N-s}b_s$
$s+1$		k_0b_0	$k_1b_0+k_0b_1$	...	$k_{j-1}b_0+k_{j-2}b_1+\ldots+k_{j-s-1}b_s$	...	$k_{N-2}b_0+\ldots+k_{N-s-2}b_s$	$l_{N-1}b_0+l_{N-s}b_1+\ldots+l_{N-s-1}b_s$
$s+2$			k_0b_0	...			⋮	
⋮								
$2s$				...	$k_{j-s}b_0+k_{j-s-1}b_1+\ldots+k_{j-2s}b_s$	...		$l_{N-s}b_0+l_{N-s-1}b_1+\ldots+l_{N-2s}b_s$
$N-1$					$k_{j-N+s+1}b_0+\ldots+k_{j-N+1}b_s$		$k_sb_0+\ldots+k_0b_s$	$l_{s+1}b_0+l_sb_1+\ldots+l_1b_s$
N					$k_{j-N+s}b_0+\ldots+k_{j-N}b_s$	...	$k_{s-1}b_0+\ldots+k_0b_{s-1}$	$l_sb_0+l_{s-1}b_1+\ldots+l_0b_s$

where $l_r=k_r+k_{r+1}+\ldots$

$$l_r = k_r + k_{r+1} + k_{r+2} + \dots \tag{5}$$

In short we have

$$p_j = \sum_{i=0}^{N} r_{ij} p_i \qquad j = 0, 1, 2, \dots, N$$

where r_{ij} are the elements in the transition matrix. We shall consider the above expression only for $j \leqq N-1$, since for $j = N$, we have from (5),

$$p_N = \frac{l_N p_0 + (l_N B_{s-1} + l_{N-1} b_s) p_1 + \dots + (l_{s-1} b_0 + l_s b_1 + \dots + l_1 b_s) p_{n-1}}{\left(1 - \sum_{r=0}^{s} l_r b_{s-r}\right)} \tag{6}$$

Therefore

$$p_j = \sum_{i=0}^{N} r_{ij} p_i \qquad (j = 0, 1, 2, \dots, N-1)$$

$$= \sum_{i=0}^{s-1} r_{ij} p_i + \sum_{i=s}^{N} r_{ij} p_i = A_1 + A_2 \quad \text{(say)} \tag{7}$$

where

$$\begin{aligned} A_1 = \sum_{i=0}^{s-1} r_{ij} p_i = k_j p_0 & \\ & + (k_j B_{s-1} + k_{j-1} b_s) p_1 \\ & + (k_j B_{s-2} + k_{j-1} b_{s-1} + k_{j-2} b_s) p_2 \\ & + \cdots \cdots \cdots \\ & + (k_j B_1 + k_{j-1} b_2 + \dots + k_{j-s+1} b_s) p_{s-1} \\ & = k_j \sum_{i=0}^{s-1} B_{s-i} p_i + \sum_{i=1}^{s-1} p_i \sum_{r=s-i+1}^{s} k_{j-i+s-r} b_r \end{aligned} \tag{8}$$

$$\begin{aligned} A_2 = \sum_{i=s}^{N} \gamma_{ij} p_i = (k_j b_0 + k_{j-1} b_1 + \dots + k_{j-s} b_s) p_s & \\ & + \cdots \cdots \cdots \\ & + (k_{j-N+s} b_0 + \dots + k_{j-N} b_s) p_N \\ & = \sum_{i=s}^{N} p_i \sum_{r=0}^{s} k_{j-i+s-r} b_r \end{aligned} \tag{9}$$

Using these values of A_1 and A_2 in (7), we have

$$p_j = k_j \sum_{i=0}^{s-1} B_{s-i} p_i + \sum_{i=1}^{s-1} p_i \sum_{r=s-i+1}^{s} k_{j-i+s-r} b_r + \sum_{i=s}^{N} p_i \sum_{r=0}^{s} k_{j-i+s-r} b_r$$

Multiplying the above expression by x^j and summing over $j = 0, 1, 2, \dots, N-1$ and using (1) and (3), we have

$$\begin{aligned} P(x) = K(x) \sum_{i=0}^{s-1} B_{s-i} p_i + \sum_{j=0}^{N-1} x^j \sum_{i=1}^{s-1} p_i \sum_{r=s-i+1}^{s} k_{j-i+s-r} b_r & \\ + \sum_{j=0}^{N-1} x^j \sum_{i=s}^{N} p^i \sum_{r=0}^{s} k_{j-i+s-r} b_r = C_1 + C_2 + C_3 \quad \text{(say)} & \end{aligned} \tag{10}$$

where

$$C_1 = K(x) \sum_{i=0}^{s-1} B_{s-i} p_i$$

$$C_2 = \sum_{j=0}^{N-1} x^j \sum_{i=1}^{s-1} p_i \sum_{r=s-i+1}^{s} k_{j-i+s-r} b_r$$

Algebraic simplification of this term gives

$$C_2 = (K(x) \cdot x^{-s} \sum_{i=1}^{s-1} \sum_{r=s-i+1}^{s} p_i b_r x^{i+r} - x^N \sum_{i=1}^{s-r-1} \sum_{r=0}^{s-2} \sum_{j=1}^{i} p_{i+r} b_{s-r} k_{N-j} x^{i-j}$$

$$= \frac{K(x)}{x^s} \sum_{i=1}^{s-1} p_i x^i \sum_{r=s-i+1}^{s} b_r x^r - x^N \cdot C^* \tag{11}$$

where

$$C^* = \sum_{r=0}^{s-2} \sum_{i=1}^{s-r-1} \sum_{j=1}^{i} p_{i+r} b_{s-r} k_{N-j} x^{i-j} = \sum_{r=0}^{s-2} \sum_{i=r+1}^{s-1} \sum_{j=1}^{i-r} p_i b_{s-r} k_{N-j} x^{i-r-j} \tag{12}$$

$$C_3 = \sum_{j=0}^{N-1} x^j \sum_{i=s}^{N} p_i \sum_{r=0}^{s} k_{j-i+s-r} b_r = \sum_{i=s}^{N} \sum_{j=i-s}^{N-1} \sum_{r=0}^{s} p_i k_{j-i+s-r} b_r x^j$$

Let $m = j - i + s - r$ (13)

Then

$$C_3 = \sum_{i=s}^{N} \sum_{m=-r}^{N-1-i+s-r} \sum_{r=0}^{s} p_i k_m b_r x^{m+i-s+r}$$

$$= \sum_{i=s}^{N} p_i x^{i-s} \sum_{m=-r}^{N-1-i+s-r} k_m x^m \sum_{r=0}^{s} b_r x^r$$

$$= \sum_{i=s}^{N} p_i x^{i-s} \sum_{r=0}^{s} \sum_{m=-r}^{N-1-i+s-r} k_m x^m \cdot b_r x^r$$

Thus

$$C_3 = \sum_{i=s}^{N} p_i x^{i-s} \sum_{r=0}^{s} b_r \sum_{m=0}^{N-1-i+s-r} x^r k_m X^m, \quad \text{since } k_j = 0 \text{ for } j < 0\,.$$

$$= \sum_{i=s}^{N} p_i x^{i-s} \sum_{r=0}^{s} b_r x^r \left[\sum_{m=0}^{N-1} k_m x^m - \sum_{m=N-i+s-r}^{N-1} k_m x^m \right]$$

$$= \sum_{i=s}^{N} p_i x^{i-s} B_s(x) \cdot K(x) - \sum_{i=s}^{N} p_i x^{i-s} \sum_{r=0}^{s} b_r \sum_{m=N-i+s-r}^{N-1} k_m x^{m+r}$$

$$= \frac{K(x)}{x^s} \sum_{i=s}^{N} p_i x^i B_s(x) - \sum_{i=s}^{N} \sum_{r=0}^{s} b_r \sum_{m=N-i+s-r}^{N-1} p_i k_m x^{i+m-s+r} \tag{14}$$

Now substituting the values of C_1, C_2 and C_3 in (10), we have

$$P(x) = K(x) \sum_{i=0}^{s-1} B_{s-i} p_i + \frac{K(x)}{x^s} \sum_{i=1}^{s-1} p_i x^i \sum_{r=s-i+1}^{s} b_r x^r - x^N \cdot C^*$$

$$+ \frac{K(x)}{x^s} \sum_{i=s}^{N} p_i x^i B_s(x) - \sum_{r=0}^{s} b_r \sum_{i=s}^{N} \sum_{m=N-i+s-r}^{N-1} p_i k_m x^{i+m-s+r}$$

$$= K(x) \sum_{i=0}^{s-1} B_{s-i} p_i + \frac{K(x)}{x^s} \left[\sum_{i=1}^{s-1} p_i x^i \sum_{r=s-i+1}^{s} b_r x^r + \sum_{i=s}^{N} p_i x^i B_s(x) \right]$$

$$- x^N \cdot C^* - \sum_{r=0}^{s} b_r \sum_{i=s}^{N} \sum_{m=N-i+s-r}^{N-1} p_i k_m x^{i+m-s+r} \tag{15}$$

Algebraic simplification of the terms in braces on the right hand side in the above expression will give us

$$P(x)=K(x)\sum_{i=0}^{s-1} B_{s-i}p_i+\frac{K(x)}{x^s}\left[\sum_{i=0}^{N} p_i x^i \sum_{r=0}^{s} b_r x^r-\sum_{i=0}^{s-1} p_i x^i \sum_{r=0}^{s-i} b_r x^r\right]$$

$$-x^N\cdot C^*-\sum_{r=0}^{s} b_r \sum_{i=s}^{N} \sum_{m=N-i+s-r}^{N-1} p_i k_m x^{m+i-s+r}$$

$$=K(x)\sum_{i=0}^{s-1} B_{s-i}p_i+\frac{K(x)}{x^s}\left[\{P(x)+p_N x^N\}B_s(x)\right.$$

$$\left.-\sum_{i=0}^{s-1} p_i x^i B_{s-i}(x)\right]-x^N\cdot C^*-\sum_{r=0}^{s} b_r \sum_{i=s}^{N} \sum_{m=N-i+s-r}^{N-1} p_i k_m x^{m+i-s+r}$$

Thus

$$P(x)\left[1-\frac{K(x)}{x^s}B_s(x)\right]=K(x)\left[\sum_{i=0}^{s-1} B_{s-i}p_i-\sum_{i=0}^{s-1} p_i x^{i-s} B_{s-i}(x)\right]$$

$$-x^N\left[C^*-p_N B_s(x)\cdot\frac{K(x)}{x^s}\right]-\sum_{r=0}^{s} b_r \sum_{i=s}^{N} \sum_{m=N-i+s-r}^{N-1} p_i k_m x^{m+i-s+r}$$

Therefore

$$P(x)=\frac{K(x)\sum_{i=0}^{s-1} p_i[B_{s-i}x^s-x^i B_{s-i}(x)]}{x^s-K(x)B_s(x)}$$

$$+x^N\left[\frac{C^* x^s-p_N B_s(x)\cdot K(x)}{K(x)B_s(x)-x^s}\right]+\frac{\sum_{i=s}^{N}\sum_{r=0}^{s}\sum_{m=N-i+s-r}^{N-1} b_r p_i k_m x^{m+i+r}}{K(x)B_s(x)-x^s}$$

where C^* is given by (12). (16)

Only the first term on the right hand side in (16) will contribute to the coefficients of x^j for $j\leqq N-1$. We disregard the second and third terms on the right hand side in (16), since they give the coefficients of x^j for $j\geqq N$. These are not needed because we wish to compare the coefficients of x^j for $j<N$ on both sides in (16) in order to evaluate p_j for $j<N$. Let

$$Q(x)=\frac{K(x)\sum_{i=0}^{s-1} p_i\{x^s B_{s-i}-x^i B_{s-i}(x)\}}{x^s-K(x)B_s(x)}$$

$$=\frac{\sum_{i=0}^{s-1} p_i\{x^s B_{s-i}-x^i B_{s-i}(x)\}}{x^s/K(x)-B_s(x)} \tag{17}$$

The function $Q(x)$ is fully determined once $p_0, p_1, \ldots, p_{s-1}$ are specified. This can be done by applying the usual arguments of analyticity of $Q(x)$ and Rouché's Theorem. The expected value of Q can only be obtained by evaluating the probabilities $p_0, p_1, \ldots, p_n$. It should be noted that the usual method of getting $E(Q)$ as $P'(1)$ can not be used since $P(X)$ is not obtained as a closed form expression.

4. Total Time Spent in the System:

Let W be the total time spent by a departing customer, who leaves Q customers behind. Let

$F(t)$ be its distribution function. Then

$$p_j = \Pr\{Q=j\} = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t) \,, \quad j<N$$

$$p_N = \Pr\{Q=N\} = \sum_{j=N}^{\infty} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t)$$

From these two expressions, we have

$$\sum_{j=0}^{N} jp_j = \sum_{j=0}^{N-1} j \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t) + N \sum_{j=N}^{\infty} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t)$$

$$\therefore \; E(Q) = \int_0^\infty \sum_{j=0}^{N-1} j e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t) + N \int_0^\infty \sum_{j=N}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t)$$

then

$$E(Q) = \int_0^\infty \sum_{j=0}^{N-1} j e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t) + N \int_0^\infty \left[1 - \sum_{j=0}^{N-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}\right] dF(t)$$

$$E(Q) = N + \int_0^\infty \sum_{j=0}^{N-1} (j-N) e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t)$$

$$= N + \int_0^\infty \left[\lambda t \sum_{j=0}^{N-2} e^{-\lambda t} \frac{(\lambda t)^j}{j!} - N \sum_{j=0}^{N-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}\right] dF(t)$$

$$= N + \int_0^\infty \left[\lambda t \frac{\Gamma(N-1, \lambda t)}{\Gamma(N-1)} - N \frac{\Gamma(N, \lambda t)}{\Gamma(N)}\right] dF(t)$$

where

$$\sum_{j=0}^{N} \frac{e^{-\lambda} \lambda^j}{j!} = \frac{\Gamma(N+1, \lambda)}{\Gamma(N+1)}$$

Thus

$$E(Q) = N + \frac{1}{\Gamma(N)} \int_0^\infty \left[(N-1)\lambda t \, \Gamma(N-1, \lambda t) - N\Gamma(N, \lambda t)\right] dF(t)$$

or

$$\int_0^\infty \left[N\Gamma(N, \lambda t) - (N-1)\lambda t \, \Gamma(N-1, \lambda t)\right] dF(t) = \Gamma(N)\left[N - E(Q)\right]$$

This expression relates the waiting time distribution and the average number of customers in the system. Thus the waiting time can be found by using the properties of incomplete gamma functions from the above expressions.

5. Special Cases*

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